$$\times \frac{St_{\mathrm{f}}}{Fo_{\mathrm{f}}}\left[\frac{k_{\mathrm{g}}(s+2)+St_{\mathrm{g}}}{s+St_{\mathrm{g}}+2}\right]\frac{2}{St_{\mathrm{g}}+2}\frac{D}{L}$$

$$+\frac{2St_{\mathrm{g}}}{s+St_{\mathrm{g}}+2}\frac{k_{\mathrm{g}}-1}{St_{\mathrm{g}}+2}\frac{\tanh\left[a_0(h_2-h_1)\right]}{a_0h_2}\frac{D}{L}\mathrm{e}^{-(s+St_{\mathrm{g}})}$$

$$+\frac{h_1}{h_2}\frac{2St_{\mathrm{g,t}}}{s+St_{\mathrm{g,t}}+2}\frac{k_{\mathrm{g}}-1}{St_{\mathrm{g,t}}+2}\frac{D}{L}$$

$$\times \mathrm{e}^{-(s+St_{\mathrm{g}})}\frac{1-\mathrm{e}^{r_0z_{\mathrm{t}}}}{r_0z_{\mathrm{t}}}\frac{St_{\mathrm{t,l}}}{n_0}(\bar{\theta}_{\mathrm{l,in}}-\bar{\theta}_{\mathrm{g,in}})$$

$$a=\sqrt{\left(\frac{1}{Fo_{\mathrm{f}}}\left(s+St_{\mathrm{f}}-\frac{St_{\mathrm{f}}\,St_{\mathrm{g}}}{s+St_{\mathrm{g}}+2}\right)\right)};$$

$$a_0=a(s=0);\quad n_0=n(s=0);\quad r_0=r(s=0)$$

$$r=\frac{w}{2Fo_{\mathrm{l}}}-\sqrt{\left(\left(\frac{w}{2Fo_{\mathrm{l}}}\right)^2+\frac{1}{Fo_{\mathrm{l}}}\left(s+St_{\mathrm{l}}-\frac{St_{\mathrm{l}}\,St_{\mathrm{t,l}}}{n}\right)\right)}$$

$$n=s+St_{\mathrm{t,l}}+St_{\mathrm{t,g}}+Fo_{\mathrm{t}}\,a\tanh\left[a(h_2-h_1)\right]-\frac{St_{\mathrm{t,g}}\,St_{\mathrm{g,t}}}{s+St_{\mathrm{g,t}}+2}$$

$$p=\left\{\frac{2St_{\mathrm{g}}}{s+St_{\mathrm{g}}+2}\frac{\tanh\left[a(h_2-h_1)\right]}{ah}+\frac{h_1}{h_2}\frac{2St_{\mathrm{g,t}}}{s+St_{\mathrm{g,t}}+2}\right\}\frac{St_{\mathrm{t,l}}}{n}\mathrm{e}^{K}$$

$$K=\frac{\delta_{\mathrm{t}}}{L}\sqrt{(s/Fo_K)};\quad Fo_K=\frac{\lambda_{\mathrm{t}}t_0}{c_{\mathrm{t}}\rho_{\mathrm{t}}L^2};\quad w=\frac{\bar{w}_{\mathrm{l}}}{\bar{w}_{\mathrm{g}}}$$

tube deflection on the liquid side

$$G_{\mathrm{u}}=\mathrm{e}^{r_{\mathrm{u}}z_{\mathrm{u}}};\quad r_{\mathrm{u}}=\frac{w}{2Fo_{\mathrm{l}}}-\sqrt{\left(\frac{w^2}{4Fo_{\mathrm{l}}^2}+\frac{1}{Fo_{\mathrm{l}}}\left(s+St_{\mathrm{l}}-\frac{St_{\mathrm{l}}\,St_{\mathrm{t,l}}}{s+St_{\mathrm{t,l}}}\right)\right)}$$

where z_{u} is the length of tube deflection (dimensionless). These transfer functions were slightly modified by comparison with a more sophisticated model derived in ref. [8]. The gas exit temperature is averaged by integrating along the y-coordinate.

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Approximate analytical solution to forced convection with arbitrary surface heat flux

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INTRODUCTION

AMBROK [1] has developed an approximate analytical solution for the local surface coefficient of heat transfer for steady flow of a fluid with arbitrary free stream velocity and density variation over a surface with an arbitrary specified wall temperature variation. This is accomplished by solving the integral form of the thermal energy equation for the local enthalpy thickness $\delta_{\mathrm{t}}^*(x)$ by use of an assertion of a universal relation, between Nusselt or Stanton number and the enthalpy thickness Reynolds number, suggested by the result for an isothermal flat plate.

Although the Stanton number expression so developed is claimed to hold for laminar, transitional, or turbulent flow, Ambrok's [1] basic, major assumption is most closely obeyed in turbulent flows. Kays and Crawford [2] give Ambrok's method high grades in the prediction of turbulent heat transfer and point out that it does well except for severely accelerated flows where values of the acceleration parameter K exceed about 0.5×10^{-6}. Data in Orlando *et al.* [3] indicate that, at least for near equilibrium flows where $K=$ constant, the local Stanton number for adverse pressure gradients is the same function of the enthalpy thickness Reynolds number as it is for a flat plate. This implies that Ambrok's method will give particularly good results in the case of adverse pressure gradients. Finally, an extension of Ambrok's result to flows with transpiration and to high speed flows is presented in Kays and Crawford [2].

In this day of computer generated finite difference solutions to forced convection heat transfer problems, the appeals of Ambrok's method are: the ease, simplicity, and economy with which it can be applied; the fact that it generally leads to an explicit, analytical expression for St_x; and the aforementioned relatively good accuracy for an extremely broad class of turbulent flow problems.

The purpose of this work is to show how Ambrok's basic ideas can be used to solve the problem of arbitrarily varying flux, q_{w}, at the wall rather than specified surface temperature variation. Specified surface heat flux situations arise in electric resistance heating, nuclear fuel rods, and often in experimental rigs where electric heaters or tape are employed. Approximate analytical expressions are developed for the Stanton number and wall temperature variation and are validated by comparison with experimental data.

ANALYSIS

Consider steady, thin boundary layer, low speed flow of a fluid with constant free stream temperature T_{s} over an arbitrarily shaped rotationally symmetric body. For these conditions, the integral form of the thermal energy equation can be written as [2]

$$\frac{\mathrm{d}\delta_{\mathrm{t}}^*}{\mathrm{d}x}+\left[\frac{1}{\rho_{\mathrm{s}}}\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}x}+\frac{1}{u_{\mathrm{s}}}\frac{\mathrm{d}u_{\mathrm{s}}}{\mathrm{d}x}+\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}x}+\frac{1}{(T_{\mathrm{w}}-T_{\mathrm{s}})}\frac{\mathrm{d}}{\mathrm{d}x}(T_{\mathrm{w}}-T_{\mathrm{s}})\right]\delta_{\mathrm{t}}^*=\frac{q_{\mathrm{w}}}{\rho_{\mathrm{s}}u_{\mathrm{s}}C_{p\mathrm{s}}(T_{\mathrm{w}}-T_{\mathrm{s}})}. \tag{1}$$

For a specified flux $q_{\mathrm{w}}(x)$ one wishes to predict $T_{\mathrm{w}}(x)$ and St_x. Hence, equation (1) is rearranged, a new dependent variable $\psi=\delta_{\mathrm{t}}^*(T_{\mathrm{w}}-T_{\mathrm{s}})$ is defined and $B(x)$ is used to denote the first three terms in the brackets of equation (1), giving

$$\frac{\mathrm{d}\psi}{\mathrm{d}x}+B(x)\psi=\frac{q_{\mathrm{w}}(x)}{\rho_{\mathrm{s}}u_{\mathrm{s}}C_{p\mathrm{s}}}. \tag{2}$$

Solving equation (2) subject to the initial condition that $\psi=\psi_0$ at $x=x_0$ yields

NOMENCLATURE

C_{ps}	specific heat of free stream
K	$(\nu/u_s^2)(\mathrm{d}u_s/\mathrm{d}x)$
L_r	reference length
Pr	Prandtl number
q_w	surface heat flux
R	local radius of revolution
Re_x	$u_s x/\nu$
St_x	Stanton number defined in equation (4)
T_s, T_w	free stream and wall temperature
ΔT	$T_w - T_s$
u_s	free stream velocity
x	position coordinate along body.

Greek symbols

β	complete Beta function
δ_t^*	$\int_0^\infty u(T-T_s)\,\mathrm{d}y/u_s(T_w-T_s)$
μ_s	dynamic viscosity
ν	μ_s/ρ_s
ξ	dummy variable of integration
ρ_s	free stream mass density.

$$\delta_t^*(T_w-T_s) = \frac{C_{ps}[\rho_s u_s R\delta_t^*(T_w-T_s)]_{x_0}+\int_{x_0}^{x} q_w(\xi)R(\xi)\,\mathrm{d}\xi}{\rho_s(x)C_{ps}u_s(x)R(x)} \tag{3}$$

where the notation $[\ ]_{x_0}$ means that every quantity in the brackets is to be evaluated at position x_0.

The local Stanton number is defined as

$$St_x = q_w(x)/\rho_s u_s C_{ps}(T_w-T_s). \tag{4}$$

In parallel with Ambrok [1] the Stanton number for a flat plate with a constant surface heat flux q_0 is

$$St_x = A\,Pr^{-2/3}\,Re_x^{m-1}. \tag{5}$$

For the flat plate case considered, $B(x) = 0$, equation (5) is inserted into equation (2) and one solves for $\psi(x)$. Next solving this for x in terms of δ_t^* and $T_w - T_s$ gives

$$x = \rho_s u_s C_{ps}\delta_t^*(T_w-T_s)/q_0. \tag{6}$$

Combining equations (4)–(6) with $\Delta T = T_w(x) - T_s$ leads to

$$\Delta T = \left(\frac{q_0}{\rho_s u_s C_{ps}}\right)^m \left(\frac{\mu_s}{\rho_s u_s}\right)^{m-1}\left(\frac{[\Delta T\delta_t^*]^{1-m}}{A\,Pr^{-2/3}}\right). \tag{7}$$

Next, as per Ambrok [1], it is assumed that equation (7) is approximately valid for arbitrary $q_w(x)$, $R(x)$, $\rho_s(x)$, and $u_s(x)$ as long as $\Delta T\delta_t^*$ is found from equation (3). Combining equations (3) and (7) and using, for turbulent flow, $m = 0.8$ and $A = 0.03$ [2], gives the general solution for ΔT. This is now used in equation (4) to yield St_x for arbitrary variation of q_w, ρ_s, R, and u_s with x

$$St_x = \frac{0.03Pr^{-2/3}\,Re_x^{-0.2}[xq_w(x)R(x)]^{0.2}}{\left\{C_{ps}[\rho_s u_s R\delta_t^*\Delta T]_{x_0}+\int_{x_0}^{x} q_w(\xi)R(\xi)\,\mathrm{d}\xi\right\}^{0.2}}. \tag{8}$$

In the event that $[\rho_s u_s R\delta_t^*\Delta T]_{x_0} = 0$, equation (8) reduces to

$$St_x = 0.03Pr^{-2/3}\,Re_x^{-0.2}\left[\frac{xq_w(x)R(x)}{\int_{x_0}^{x} q_w(\xi)R(\xi)\,\mathrm{d}\xi}\right]^{0.2}. \tag{9}$$

The expressions for the wall temperature variation with x caused by the flux $q_w(x)$ are found by simply combining equation (8) or (9) with equation (4) to eliminate St_x. For the general case, equation (8), this gives

$$\Delta T = \frac{33.33q_w^{0.8}x^{0.8}}{k\,Pr^{1/3}\,Re_x^{0.8}} \times \left[\frac{C_{ps}[\rho_s u_s R\delta_t^*\Delta T]_{x_0}+\int_{x_0}^{x} q_w(\xi)R(\xi)\,\mathrm{d}\xi}{R(x)}\right]^{0.2}. \tag{10}$$

The effect of temperature dependent property variation *across* the boundary layer, in the general relations, equations (8) and (10), can be approximately accounted for by employing the temperature ratio correction scheme advanced in ref. [2].

In order to check and validate the relations developed, they are next applied to some representative forced convection problems and comparison is made to either the accepted analytical solution, if available, or to experimental data.

Power law flux variation

For an unheated starting length x_0 beyond which the surface heat flux varies as bx^n, the solution for St_x in equation (8) is

$$q_w = 0,\ x < x_0 \qquad St_x = \frac{0.03Pr^{-2/3}\,Re_x^{-0.2}(n+1)^{0.2}}{[1-(x_0/x)^{n+1}]^{0.2}}$$
$$= bx^n,\ x > x_0 \tag{11}$$

The accepted analytical solution for the flat plate with no unheated starting length can be found, from a relation given in ref [2], as

$$St_x = \frac{0.26316Pr^{-2/3}\,Re_x^{-0.2}}{\beta[\frac{1}{9}, \frac{10}{9}(n+1)]} \tag{12}$$

where the denominator is the beta function. (Kays, in ref. [2] as well as in his other works, found that a Prandtl number dependence in the Stanton number of $Pr^{-0.4}$ rather than $Pr^{-2/3}$ does a better job of predicting the data for Prandtl numbers close to those of air.)

Upon comparing the present result, equation (11) with $x_0 = 0$, to the accepted relation, equation (12), one finds that equation (11) is up to 8% higher than equation (12) for a wide range of n between -0.8 and 2.0 and is exact at $n = 0$, constant flux, and $n = -0.2$, constant wall temperature. These results at $n = 0$ and -0.2 are expected because of Ambrok's method being based on the accepted result for the flat plate at constant flux.

Orlando *et al.* [3], in one of the experiments reported there, deal with a strongly decelerating flow in which $u_s \sim x^{-0.275}$ and $\Delta T =$ constant. For these conditions, it can be verified from equation (10) that $q_w(x) \sim x^{-0.475}$ and that equation (11) gives

$$St_x = 0.02637Pr^{-0.4}\,Re_x^{-0.2} \tag{13}$$

where their suggested Prandtl number dependence has been incorporated in equation (13) rather than in equation (11). In Fig. 1, equation (13), shown as a solid line, is compared to their experimental data, the circles, and it is evident that agreement is very good.

A more complex situation is given in Run 27 of Moretti and Kays [4]. This case of mild acceleration ($K = 0.2\times10^{-6}$) is characterized by the following conditions: $0 \leqslant x < x_1$, $u_s =$ constant, $\Delta T = \Delta T_1$, also a constant; for $x_1 < x \leqslant x_2$,

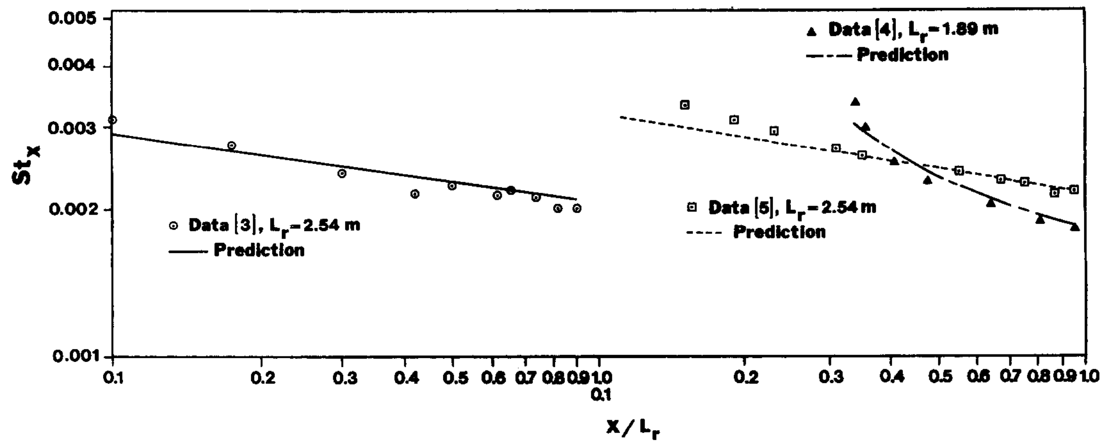

FIG. 1. Local Stanton number predictions and comparison to experimental data.

u_s = constant, $\Delta T = \Delta T_2$, another constant; and when $x > x_2$, $u_s \sim x^{0.3575}$ and $\Delta T_3 = \Delta T_2$, the same value held between x_1 and x_2.

This case, like the previous one dealt with just above from ref. [3], is a case of specified wall temperature and therefore is more easily and directly handled using Ambrok's [1] original result. Here we attempt to treat it as a specified surface heat flux problem as a check of our result, equation (8). This was accomplished as follows. For the first region where $0 \leqslant x \leqslant x_1$, $\Delta T = \Delta T_1 \sim x^0$ since ΔT is constant. Using this in equation (10) with a power law flux variation $q_w = b_1 x^{n_1}$ shows that $n_1 = -0.2$. Using this in equation (11) gives St_x for $0 \leqslant x \leqslant x_1$.

In the second region, $x_1 < x < x_2$, $\Delta T = \Delta T_2$. Again $\Delta T \sim x^0$ and if one assumes a flux variation of $q_w = b_2 x^{n_2}$ in this region, equation (10) indicates, as long as x is not too close to x_1, that choosing $n_2 = -0.2$ yields a constant ΔT in this region. The ratio of b_2/b_1 is then found by setting the known value of $\Delta T_2/\Delta T_1$ equal to the expression for this ratio as given by the right-hand side of equation (10) for the two different regions. This gave $b_2/b_1 = (\Delta T_2/\Delta T_1)^{1.25}$. With this, equation (8) becomes

$$St_x = \frac{0.0287 Pr^{-0.4} Re_x^{-0.2}}{\{1-[1-(\Delta T_1/\Delta T_2)^{1.25}](x_1/x)^{0.8}\}^{0.2}} \tag{14}$$

for $x_1 \leqslant x \leqslant x_2$.

For the third region, $x > x_2$, $\Delta T_3 = \Delta T_2$ and $u_s \sim x^{0.3575}$. It is assumed that the flux variation in this region can be represented by $q_w = b_3 x^{n_3}$, at least for x not too close to x_2. Following the same procedure described above for the second region, one finds that n_3 must be 0.086 and b_3/b_2 is found by the condition that $\Delta T_2 = \Delta T_3$. With this along with $n_1 = n_2$ and b_2/b_1 from the second region and using the different flux expressions for the three different regions in the integral of equation (8), we get for $x > x_2$

$$St_x = \frac{0.0305 Pr^{-0.4} Re_x^{-0.2}}{\left\{1-\left(\frac{x_2}{x}\right)^{1.086}+1.3575\left[\frac{b_2}{b_3}\left(\frac{x_2^{0.8}-x_1^{0.8}}{x^{1.086}}\right)+\frac{b_1}{b_3}\frac{x_1^{0.8}}{x^{1.086}}\right]\right\}^{0.2}}. \tag{15}$$

In both equations (14) and (15), the power used on the Prandtl number is the one suggested in ref. [4] rather than the one which appears in the general result, equation (8). Figure 1 compares the predictions of equations (14) and (15) to the data, the triangles, of ref. [4] for Run 27 (no experimental data were presented for $x < x_1$). As is seen from the figure there is good agreement between the predictions and data.

Orlando *et al.* [5] present data for a situation involving mild deceleration with $u_s \sim x^{-0.15}$ and a variable wall temperature. A curve fit to the ΔT data in ref. [5] gives $\Delta T \sim x^{0.3345}$. With these x dependencies for u_s and ΔT inserted into equation (10), one finds that $q_w \sim x^{0.0145}$. Thus, equation (8) reduces to

$$St_x = 0.0301 Pr^{-0.4} Re_x^{-0.2}. \tag{16}$$

Comparing equation (16) with the data, the squares in Fig. 1, it is seen that agreement is very good except at smaller non-dimensional distances x/L_r. This may be due to the fact that the author could not determine, from Fig. 7 in ref. [5], what wall temperature variation was used in the experiment between $x/L_r = 0$ and 0.73.

Case II of McCarthy and Hartnett [6] is a linearly accelerated flow with an average K of about 0.4×10^{-6} with a long unheated starting length preceding a constant flux test section. Hence, the present work gives the Stanton number solution as being equation (11) with $n = 0$. However, it is known, ref. [4] for example, that Ambrok's method is suspect in the region just downstream of an unheated starting length. This was borne out by our predictions which were considerably too high over much of the heated length since, in this case, the heated length was only slightly longer than the unheated starting length. Moretti and Kays [4] suggest a modification of Ambrok's method for specified wall temperature variation that will give it more nearly the correct dependence on the unheated starting length x_0. Basically this involves replacing the power 0.2 on the term associated only with the unheated starting length, namely the denominator of equation (11), by the power 0.12. When this was done, the experimental Stanton number data of Case II in ref. [6] was predicted within 5% and was always higher than the data.

Thus, based upon the results of this calculation with an unheated starting length and the comparison of predictions with data in ref. [6], and guided also by the suggestions in ref. [4], it is recommended that equation (11) of this paper be used with a power of 0.12 *in the denominator* rather than the power 0.2.

CONCLUDING REMARKS

Easy to use, approximate analytical expressions for the wall temperature and Stanton number variation with position have been derived for the case of arbitrary surface heat flux distribution. This was accomplished using the same type of approach as was employed by Ambrok [1] for arbitrary surface temperature distribution.

The forms taken by the general expressions for the case of power law surface heat flux variation were derived. These approximate forms were compared to some available accepted analytical solutions, and to experimental data for a number of different decelerating and accelerating flows with variable surface flux, as a check and a demonstration of expected accuracy of the approximate solutions.

One of the comparisons of predictions to experimental data resulted in the suggestion that an empirical modification be made to equation (11), namely the use of the power 0.12 instead of 0.20 in its denominator.

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Negative heat transfer in separated flows

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1. INTRODUCTION

More often than not, time-dependent flows reveal unexpected instantaneous features couched under the time averaged. A classic case is that of the time-dependent, large-scale coherent structure found in a turbulent mixing layer [1], a flow regarded disorderly and featureless in time-averaged measurements. As another and more recent example, it was pointed out [2] that the vortex street behind a body can separate the total temperatures into hot and cold spots around vortices, although the steady data only disclose the colder wake. The latter is a time-averaged 'footprint' left by the former, the instantaneous total temperature separation; the time-averaged, mean colder wake manifests itself as the negative recovery factor on the rearward portion of the cylinder surface, an effect first discovered by Eckert and Weise [3] for a thermally insulated cylinder in air.

For a heat-conducting cylinder, then, even when the bulk fluid temperature is warmer than the surface temperature, the heat can be transferred locally from the rearward surface to the adjacent fluid. It is, therefore, contrary to the conventional expectation that convective heat transfer takes place in the same direction as the surface to bulk fluid temperature difference. Such '*negative*' heat transfer, obviously a time-averaged effect, conceals once again the more intricate transient. Viewed from the point of the time-dependent flow, transient heat transfer is found to fluctuate between *negative* and *positive* values, leaving an imprint of the negative heat transfer as its mean; the mechanism is subtler than to be superficially expected from the aforementioned instantaneous total temperature separation. This is the subject to be discussed below.

2. DISCUSSION AND COMPUTATIONAL RESULTS

For the present purpose, it is of course convenient to investigate the instantaneous *static* temperature distribution rather than the total temperature treated in ref. [2]. Consider a cylinder held at uniform wall temperature (T_w) immersed in a compressible fluid, the upstream static temperature (T_∞) of which is higher than T_w. (Therefore, the upstream total temperature ($T_{t\infty}$) is also higher than T_w.)

In the forward portion of the cylinder where the flow is attached, the heat always transfers from the warmer fluid to the colder surface. Consider, however, the rearward portion where the boundary layer is separating from one side and a vortex is being formed (Fig. 1). As the vortex rolls up and gains strength, the static pressure at the vortex center continues to fall; the pressure drop is induced to counterbalance

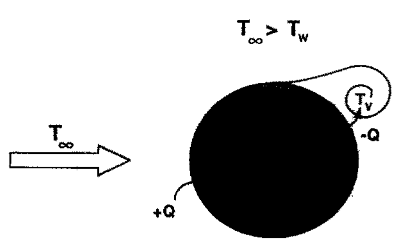

Fig. 1. Sketch of negative heat transfer.